Math 33A Worksheet 8 Solutions

Exercise 1. True or false:

(a) If A is orthogonal then it is invertible.

(b) If A is symmetric it is invertible.

(c) Let V be a subspace of $\mathbb{R}^n$ with orthonormal basis $\{u_1, \ldots, u_m\}$, and let $\{v_1, \ldots, v_{n-m}\}$ be an orthonormal basis for $V^\perp$. Then $\{u_1, \ldots, u_m, v_1, \ldots, v_{n-m}\}$ is an orthonormal basis for $\mathbb{R}^n$.

(d) The entries of an orthogonal matrix are all less than or equal to 1 in absolute value.

(e) Let V be a subspace of $\mathbb{R}^n$ and B the matrix for orthogonal projection onto V. Then $B^2 = B$.

(f) Let $\mathscr{B} = \{v_1, v_2, v_3\} = \left\{ \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \right\}$ be an ordered basis for $\mathbb{R}^3$. Then $\begin{bmatrix} 1 \\ -8 \\ 3 \end{bmatrix}_{\mathscr{B}} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$.

(g) If $v_1, \ldots, v_m$ is a basis of unit length vectors for a subspace V, there is an orthonormal basis of V containing the vectors v_1 and v_2.

(h) For all $v, w \in \mathbb{R}^n$, $\langle v, w\rangle^2 \leq ||v||^2||w||^2$ with equality if and only if v, w are perpendicular. **Notation:** $\langle v, w\rangle$ refers to the dot product $v \cdot w$.

(a) True. Since A is orthogonal, A^{-1} exists and is equal to A^T.

(b) False. $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is symmetric but not invertible.

(c) True. Each of $u_1, \ldots, u_m, v_1, \ldots, v_{n-m}$ is unit length, and $u_i \cdot u_j = 0$ for $i \neq j$, $v_i \cdot v_j = 0$ for $i \neq j$, and $u_i \cdot v_j = 0$ for all i, j since $u_i \in V$ and v_j is in the perpendicular subspace $V^\perp$. Thus, the $u_1, \ldots, u_m, v_1, \ldots v_{n-m}$ are orthogonal

(d) True. Let $A = \begin{bmatrix} u_1 & \ldots & u_n \end{bmatrix}$ be orthogonal, so $\{u_1, \ldots, u_n\}$ are orthonormal. Therefore for each of the u_i with entries $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, we have $1 = ||u_i|| = \sqrt{x_1^2 + \cdots + x_n^2}$, so $x_j^2 \leq 1$ for all j, so all the elements of A are bounded by 1 in absolute value.

(e) True. For any $v \in \mathbb{R}^n$, $B^2v = B(Bv)$. Since $\mathrm{Im}B = V$ and B fixes V since B is orthogonal projection on to V, $B(Bv) = Bv$. Therefore, since $B^2v = Bv$ for all $v \in \mathbb{R}^n$, $B^2 = B$.

(f) No, since $\begin{bmatrix} 1 \\ -8 \\ 3 \end{bmatrix}_{\mathscr{B}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ satisfies $av_1 + bv_2 + cv_3 = \begin{bmatrix} 1 \\ -8 \\ 3 \end{bmatrix}$, but

$$2v_1 + v_2 - 2v_3 = \begin{bmatrix} 2 \\ -5 \\ 5 \end{bmatrix} \neq \begin{bmatrix} 1 \\ -8 \\ 3 \end{bmatrix}$$

(g) False. Consider

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

which are unit length and form a basis for $\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rangle$. Since $v_1 \cdot v_2 \neq 0$, we cannot form an orthonormal basis with v_1, v_2.

(h) False. If "perpendicular" was replaced with "parallel", this is true and is the Cauchy Schwarz inequality. But if v, w are perpendicular, then $v \cdot w = 0$ and thus $v \cdot w \neq ||v||^2||w||^2$ for any non-zero perpendicular v, w.

Exercise 2. Let

$$A = \begin{bmatrix} 2 & 3 & -1 & 0 \\ 2 & 1 & 1 & -4 \end{bmatrix}$$

(a) Find an orthonormal basis $\mathscr{B} = \{u_1, u_2\}$ for $\ker A$.

(b) Using your basis from part (a), find the matrix B for orthogonal projection onto $\ker A$.

(c) Find $B_{\mathscr{B}'}$. **Typo**: $\mathscr{B}'$ should be an orthonormal basis of $\mathbb{R}^4$ containing $\mathscr{B}$. So it should be an orthonormal basis $\{u_1, u_2, v_1, v_2\}$ for v_1, v_2 a basis for $\ker A^\perp$. It doesn't matter what v_1, v_2 you choose as long as $\mathscr{B}'$ is orthonormal.

(d) **(Challenge).** Generalize your observation from part (c). Given an orthonormal basis $\mathscr{B} = \{u_1, \ldots, u_m\}$ for a subspace V and an extension to an orthonormal basis $\mathscr{B}' = \{u_1, \ldots, u_m, v_1, \ldots, v_{n-m}\}$ of $\mathbb{R}^n$, what is the matrix for orthogonal projection onto V in the basis $\mathscr{B}'$?

(a) RREF: $\begin{bmatrix} 1 & 0 & 1 & -3 \\ 0 & 1 & -1 & 2 \end{bmatrix}$, which results in $\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix} \}$ being a basis for $\ker A$. Performing

Gram-Schmidt with $v_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix}$, we have $v_1^\perp = v_1$ and

$$v_2^\perp = v_2 - \frac{(v_1^\perp \cdot v_2)}{||v_1^\perp||^2} v_1^\perp = v_2 + \frac{5}{3} v_1^\perp = \begin{bmatrix} 4/3 \\ -1/3 \\ 5/3 \\ 1 \end{bmatrix}$$

Thus, we get

$$u_1 = v_1^\perp / ||v_1^\perp|| = \begin{bmatrix} \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \end{bmatrix}, u_2 = v_2^\perp / ||v_2^\perp|| = \begin{bmatrix} 4/\sqrt{51} \\ -1/\sqrt{51} \\ 5/\sqrt{51} \\ 3/\sqrt{51} \end{bmatrix}$$

(b) Let $Q = \begin{bmatrix} u_1 & u_2 \end{bmatrix}$. Then $B = QQ^T$, so

$$B = \begin{bmatrix} \frac{-1}{\sqrt{3}} & 4/\sqrt{51} \\ \frac{1}{\sqrt{3}} & -1/\sqrt{51} \\ \frac{1}{\sqrt{3}} & 5/\sqrt{51} \\ 0 & 3/\sqrt{51} \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ 4/\sqrt{51} & -1/\sqrt{51} & 5/\sqrt{51} & 3/\sqrt{51} \end{bmatrix} = \frac{1}{17} \begin{bmatrix} 11 & -7 & 1 & 4 \\ -3 & 6 & 4 & -1 \\ -7 & 4 & 14 & 5 \\ 4 & -1 & 5 & 3 \end{bmatrix}$$

(c) Notice that $Bu_1 = u_1$ since $u_1 \in \ker A$, and similarly $Bu_2 = u_2$. Since v_1, v_2 are perpendicular to $\ker A$, $Bv_1 = Bv_2 = 0$. Therefore with respect to this basis,

$$B_{\mathscr{B}'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(d) The matrix will always look like the one above, with m ones along the diagonal and $n - m$ zeroes everywhere else.

Exercise 3. Find the QR decomposition of the matrix

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

First we perform Gram Schmidt on the columns resulting in the vectors

$$u_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

Thus, $Q = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}$. Notice that R is a matrix which satisfies $A = QR$, so $Q^{-1}A = R$. Since Q is orthogonal we have $Q^{-1} = Q^T$, so $R = Q^T A$. Thus, we have

$$R = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & -\sqrt{2} & -\frac{1}{\sqrt{2}} \\ 0 & 2 & 1 \\ 0 & 0 & \frac{3}{\sqrt{2}} \end{bmatrix}$$

Exercise 4. Let $A = \begin{bmatrix} 3 & 2 & 2 & 1 \\ 0 & -1 & 2 & 1 \\ 1 & 4 & -6 & -3 \end{bmatrix}$, $V = \operatorname{im} A$.

(a) Find the projection matrix B for proj_V **(Typo: V, not W)**, projection onto V.

(b) Using B, determine whether $\begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} \in \operatorname{im} A$ (since B is projection onto V, a vector v is in V if and only if $B \cdot v = v$).

(c) Find the least squares solution to $Ax = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$.

(a) RREF:

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus columns 1 and 2 form a basis for ImA, so $v_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ form a basis for V. We now perform Gram Schmidt with $v_1^\perp = v_1$ and

$$v_2^\perp = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$$

Therefore, $u_1 = v_1^\perp/||v_1^\perp|| = \begin{bmatrix} 3/\sqrt{10} \\ 0 \\ 1/\sqrt{10} \end{bmatrix}$, $u_2 = v_2^\perp/||v_2^\perp|| = \begin{bmatrix} -1/\sqrt{11} \\ -1/\sqrt{11} \\ 3/\sqrt{11} \end{bmatrix}$ are an orthonormal basis for V. Thus letting $Q = \begin{bmatrix} u_1 & u_2 \end{bmatrix}$ we have that $B = QQ^T$, so

$$B = \begin{bmatrix} 3/\sqrt{10} & -1/\sqrt{11} \\ 0 & -1/\sqrt{11} \\ 1/\sqrt{10} & 3/\sqrt{11} \end{bmatrix} \begin{bmatrix} 3/\sqrt{10} & 0 & 1/\sqrt{10} \\ -1/\sqrt{11} & -1/\sqrt{11} & 3/\sqrt{11} \end{bmatrix} = \frac{1}{110} \begin{bmatrix} 109 & 10 & 3 \\ 10 & 10 & -30 \\ 3 & -30 & 101 \end{bmatrix}$$

(b) This vector is in ImA if and only if $Bv = v$. Thus, we compute

$$B \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} = \frac{1}{110} \begin{bmatrix} 319 \\ -110 \\ 473 \end{bmatrix} \neq \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$$

(c) The least squares solution to $Ax = v$ is given by the solution x to $Ax = Bv$. We computed Bv above, so solving for $Ax = \frac{1}{110} \begin{bmatrix} 319 \\ -110 \\ 473 \end{bmatrix}$ we find:

$$\left(\begin{array}{cccc|c} 3 & 2 & 2 & 1 & 319/110 \\ 0 & -1 & 2 & 1 & -1 \\ 1 & 4 & -6 & -3 & 473/110 \end{array}\right) \Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 3/10 \\ 0 & 1 & -2 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

So there are multiple least squares solutions, one given by $\begin{bmatrix} 3/10 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.